THE VARIATIONAL THEORY OF THE PERFECT DILATON-SPIN FLUID IN A WEYL–CARTAN SPACE

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Abstract

The variational theory of the perfect fluid with intrinsic spin and dilatonic charge (dilaton-spin fluid) is developed. The spin tensor obeys the classical Frenkel condition. The Lagrangian density of such fluid is stated, and the equations of motion of the fluid, the Weyssenhoff-type evolution equation of the spin tensor and the conservation law of the dilatonic charge are derived. The expressions of the matter currents of the fluid (the canonical energy-momentum 3-form, the metric stress-energy 4-form and the dilaton-spin momentum 3-form) are obtained.

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1. INTRODUCTION

The basic concept of the modern fundamental physics consists in proposition that spacetime geometrical structure is compatible with the properties of matter filling the spacetime. As a result of this fact the matter dynamics exhibits the constraints on a metric and a connection of the spacetime manifold. We shall consider the spacetime with the Weyl–Cartan geometry and the perfect dilaton-spin fluid as matter that fills the spacetime, generates the spacetime Weyl–Cartan geometrical structure and interacts with it.

The perfect dilaton-spin fluid is a perfect fluid, every particle of which is endowed with intrinsic spin and dilatonic charge. This model of the fluid, on the one hand, generalizes the Weyssenhoff–Raabe perfect spin fluid model[1] and, on the other hand, can be considered as a particular case of the model of the perfect hypermomentum fluid in a metric-affine space (see Ref. 2 and references therein). Significance of matter with dilatonic charge is based on the fact that a low-energy effective string theory is reduced to the theory of interacting metric and dilatonic field.[3] In this connection the dilatonic gravity is one of the attractive approaches to the modern gravitational theory.

The theory of the perfect fluid with intrinsic degrees of freedom being developed, the additional intrinsic degrees of freedom of fluid particles are described with the help of a material frame formed by four vectors $\bar{l}_p$ ($p = 1, 2, 3, 4$) (called 'directors'), adjoined with each element of the fluid. Three of the directors ($p = 1, 2, 3$) are space-like and the fourth ($p = 4$) is time-like. In some theories it is accepted that the time-like director is collinear to the 4-velocity of the fluid element. In this case the fields of the directors form the frame of reference comoving with fluid (the rest frame of reference for each fluid element).

The spin intrinsic degrees of freedom are characterized by the spin tensor. It is well-known due to Frenkel[4] that the spin tensor of a particle is spacelike in its nature that is the fact of fundamental physical meaning. This leads to the classical Frenkel condition for a fluid element, $S_{pq}u^q = 0$, where u^q is the 4-velocity of the fluid element with respect to the frame of reference formed by the directors and S_{pq} is the specific (per particle) spin

tensor of the fluid element with respect to the same frame, which is calculated by some averaging procedure over the all particles contained into the fluid element. The Frenkel condition in this form is one of the main principle that underlies the Weyssenhoff spin fluid theory.[1] Validity of this condition defines the range of applicability of the Weyssenhoff theory. It should be mentioned that the Frenkel condition appears to be a consequence of the generalized conformal invariance of the Weyssenhoff perfect spin fluid Lagrangian.[5]

We develop the variational theory of the perfect dilaton-spin fluid in a Weyl–Cartan space Y_4 using the exterior form language according to Trautman.[6] This theory generalizes the variational theory of the Weyssenhoff–Raabe perfect spin fluid based on accounting the constraints in the Lagrangian density of the fluid with the help of Lagrange multipliers (see Ref. 2 and references therein).

2. THE DYNAMICAL VARIABLES AND CONSTRAINTS

Let us consider a connected 4-dimensional oriented differentiable manifold $\mathcal{M}$ equipped with a linear connection Γ, a metric g of index 1 and volume 4-form η,

$$\eta = \frac{1}{4!}\sqrt{-g}\epsilon_{\alpha\beta\sigma\rho}\theta^{\alpha} \wedge \theta^{\beta} \wedge \theta^{\sigma} \wedge \theta^{\rho} \; , \qquad g = \det\|g_{\alpha\beta}\| \; . \tag{2.1}$$

Here $\epsilon_{\alpha\beta\sigma\rho}$ are the components of the totaly antisymmetric Levi-Civita 4-form density ($\epsilon_{1234} = -1$). We use a local vector frame $\bar{e}_{\beta}$ ($\beta = 1, 2, 3, 4$) being nonholonomic in general and a 1-form coframe θ^{α} with $\bar{e}_{\beta}\rfloor\theta^{\alpha} = \delta^{\alpha}_{\beta}$ ($\rfloor$ means the interior product). We shall use according to Trautman[6] 3-form fields η_{α} and 2-form fields $\eta_{\alpha\beta}$,

$$\begin{aligned}
&\eta_{\alpha} = \bar{e}_{\alpha}\rfloor\eta = *\theta_{\alpha} \; , \qquad \eta_{\alpha\beta} = \bar{e}_{\beta}\rfloor\eta_{\alpha} = *(\theta_{\alpha} \wedge \theta_{\beta}) \; , \\
&\theta^{\sigma} \wedge \eta_{\alpha} = \delta^{\sigma}_{\alpha}\eta \; , \qquad \theta^{\sigma} \wedge \eta_{\alpha\beta} = -2\delta^{\sigma}_{[\alpha}\eta_{\beta]} \; ,
\end{aligned} \tag{2.2}$$

where $*$ is the Hodge dual operator.

A Weyl–Cartan space Y_4 is a space with a curvature 2-form $\mathcal{R}^{\alpha}{}_{\beta}$, a torsion 2-form $\mathcal{T}^{\alpha}$ and with the metric g and the connection Γ which obey the constraint,

$$-\mathcal{D}g_{\alpha\beta} =: \mathcal{Q}_{\alpha\beta} = \frac{1}{4}g_{\alpha\beta}\mathcal{Q}\ , \qquad \mathcal{Q} := g^{\alpha\beta}\mathcal{Q}_{\alpha\beta} = Q_\alpha\theta^\alpha\ , \tag{2.3}$$

where $\mathcal{Q}_{\alpha\beta}$ is a nonmetricity 1-form, $\mathcal{Q}$ is a Weyl 1-form and $\mathcal{D} := d + \Gamma\wedge\ldots$ is the covariant exterior differential with respect to the connection 1-form $\Gamma^\alpha{}_\beta$.

Each fluid element possesses a 4-velocity vector $\bar{u} = u^\alpha\bar{e}_\alpha$ which is corresponded to a flow 3-form u, $u := \bar{u}\rfloor\eta = u^\alpha\eta_\alpha$ and a velocity 1-form $*u = u_\alpha\theta^\alpha = g(\bar{u},)$ with

$$*u\wedge u = -c^2\eta\ , \tag{2.4}$$

that means the usual condition $g(\bar{u},\bar{u}) = -c^2$, where $g(,)$ is the metric tensor.

In the exterior form language the material frame of the directors turns into the coframe of 1-forms l^p $(p = 1,2,3,4)$, which have dual 3-forms l_q, while the constraint,

$$l^p\wedge l_q = \delta^p_q\eta\ , \tag{2.5}$$

being fulfilled. This constraint means that

$$l^p_\alpha l^\alpha_q = \delta^p_q\ , \qquad l^p_\alpha l^\beta_p = \delta^\beta_\alpha\ ,$$

where the component representations were introduced, $l^p = l^p_\alpha\theta^\alpha$, $l_q = l^\beta_q\eta_\beta$.

The perfect dilaton-spin fluid obeys the Frenkel condition, which can be expressed in two forms,

$$S^p{}_q u^q = 0\ , \qquad u_p S^p{}_q = 0\ . \tag{2.6}$$

Here $u^q = u^\alpha l^q_\alpha$, $u_p = u_\alpha l^\alpha_p$. The Frenkel conditions (2.6) are equivalent to the equality,

$$\Pi^p_r\Pi^t_q S^r{}_t = S^p{}_q\ , \quad \Pi^p_r := \delta^p_r + \frac{1}{c^2}u^p u_r\ , \tag{2.7}$$

where Π^p_r is the projection tensor which separates the subspace orthogonal to the fluid velocity. The equality (2.7) means that the spin tensor $S^p{}_q$ is spacelike.

In the exterior form language the Frenkel conditions (2.6) can be written as

$$S^p{}_q l_p\wedge *u = 0\ , \qquad S^p{}_q l^q\wedge u = 0\ . \tag{2.8}$$

In case of the dilaton-spin fluid the spin dynamical variable of the Weyssenhoff fluid is generalized and becomes the new dynamical variable named the dilaton-spin tensor $J^p{}_q$:

$$J^p{}_q = S^p{}_q + \frac{1}{4}\delta^p_q J \ , \qquad S^p{}_q := J^{[p}{}_{q]} \ , \qquad J := J^p{}_p \ . \tag{2.9}$$

It is important that only the first term of $J^p{}_q$ (the spin tensor) obeys to the Frenkel condition. The second term is proportional to the specific (per particle) dilatonic charge J of the fluid element. The existence of the dilaton charge is the consequence of the extension of the Poincaré symmetry (with the spin tensor as the dynamical invariant) to the Poincaré–Weyl symmetry with the dilaton-spin tensor as the dynamical invariant.

A fluid element moving, the fluid particles number conservation law (the conservation of the baryon number),[7] and the entropy conservation law are fulfilled

$$d(nu) = 0 \ , \qquad d(nsu) = 0 \ , \tag{2.10}$$

where n is the fluid particles concentration equal to the number of fluid particles per a volume unit, and s is the the specific (per particle) entropy of the fluid, both in the rest frame of reference.

The measure of intrinsic motion contained in a fluid element is the quantity $\Omega^q{}_p$ which generalizes the intrinsic 'angular velocity' of the Weyssenhoff spin fluid theory,[1]

$$\Omega^q{}_p \eta := u \wedge l^q_\alpha \mathcal{D} l^\alpha_p \ , \tag{2.11}$$

where $\mathcal{D}$ means the exterior covariant differential with respect to the connection 1-form $\Gamma^\alpha{}_\beta$,

$$\mathcal{D} l^\alpha_p = dl^\alpha_p + \Gamma^\alpha{}_\beta l^\beta_p \ . \tag{2.12}$$

An element of the perfect dilaton-spin fluid possesses the additional intrinsic 'kinetic' energy density 4-form,

$$E = \frac{1}{2} n J^p{}_q \Omega^q{}_p \eta = \frac{1}{2} n S^p{}_q u \wedge l^q_\alpha \mathcal{D} l^\alpha_p + \frac{1}{8} n J u \wedge l^p_\alpha \mathcal{D} l^\alpha_p \ . \tag{2.13}$$

The internal energy density of the fluid ε depends on the extensive (additive) thermodynamic parameters n, s (entropy of a fluid element per particle), $S^p{}_q$, J and obeys to the first thermodynamic principle,

$$d\varepsilon(n, s, S^p{}_q, J) = \frac{\varepsilon + p}{n} dn + nT ds + \frac{\partial \varepsilon}{\partial S^p{}_q} dS^p{}_q + \frac{\partial \varepsilon}{\partial J} dJ \ , \tag{2.14}$$

where p is the hydrodynamic fluid pressure, the fluid particles number conservation law (2.10) being taken into account.

3. THE LAGRANGIAN DENSITY AND THE EQUATIONS OF MOTION OF THE FLUID

The perfect fluid Lagrangian density 4-form of the perfect dilaton-spin fluid should be chosen as the remainder after subtraction the internal energy density of the fluid ε from the 'kinetic' energy (2.13) with regard to the constraints (2.4), (2.8), (2.10) which should be introduced into the Lagrangian density by means of the Lagrange multipliers λ, χ^q, ζ_p, φ and τ, respectively. As a result of the Sec. 2 the Lagrangian density 4-form has the form,

$$\begin{aligned} \mathcal{L}_m = L_m \eta = &-\varepsilon(n, s, S^p{}_q, J)\eta + \frac{1}{2} n S^p{}_q u \wedge l^q_\alpha \mathcal{D} l^\alpha_p + \frac{1}{8} n J u \wedge l^p_\alpha \mathcal{D} l^\alpha_p \\ &+ nu \wedge d\varphi + n\tau u \wedge ds + n\lambda(*u \wedge u + c^2 \eta) + n\chi^q S^p{}_q l_p \wedge *u + n\zeta_p S^p{}_q l^q \wedge u \ . \end{aligned} \tag{3.1}$$

The fluid motion equations and the evolution equation of the dilaton-spin tensor are derived by the variation of (3.1) with respect to the independent variables n, s, $S^p{}_q$, J, u, l^q and the Lagrange multipliers, the thermodynamic principle (2.14) being taken into account. We shall consider the 1-form l^q as an independent variable and the 3-form l_p as a function of l^q by means of (2.5). As a result of such variational machinery one gets the constraints (2.4), (2.8), (2.10) and the following variational equations,

$$\delta n : \quad \frac{1}{2} n S^p{}_q u \wedge l^q_\alpha \mathcal{D} l^\alpha_p + \frac{1}{8} n J u \wedge l^p_\alpha \mathcal{D} l^\alpha_p + nu \wedge d\varphi = (\varepsilon + p)\eta \ , \tag{3.2}$$

$$\delta s : \quad T\eta + u \wedge d\tau = 0 \ , \tag{3.3}$$

$$\delta S^p{}_q : \quad \frac{\partial \varepsilon}{\partial S^p{}_q}\eta = \frac{1}{2} n \Omega^{[q}{}_{p]}\eta + n\chi^{[q} l_{p]} \wedge *u + n\zeta_{[p} l^{q]} \wedge u \ , \tag{3.4}$$

$$\delta J : \quad \frac{\partial \varepsilon}{\partial J}\eta = \frac{1}{8} n \Omega^p{}_p \eta \ , \tag{3.5}$$

$$\delta u : \quad d\varphi + \tau ds - 2\lambda * u + \chi^q S_{\beta q}\theta^\beta - \zeta_p S^p{}_q l^q$$

$$+\frac{1}{2}S^{p}{}_{q}l^{q}_{\alpha}\mathcal{D}l^{\alpha}_{p}+\frac{1}{8}Jl^{p}_{\alpha}\mathcal{D}l^{\alpha}_{p}=0\ , \tag{3.6}$$

$$\delta l^{q}:\quad \frac{1}{2}\dot{S}^{\sigma}{}_{\rho}l^{\rho}_{q}\eta_{\sigma}+\frac{1}{8}\dot{J}l_{q}-\chi^{r}S^{p}{}_{r}u_{q}l_{p}-\zeta_{r}S^{r}{}_{q}u=0\ . \tag{3.7}$$

Here the 'dot' notation for the tensor object Φ is introduced, $\dot{\Phi}^{\alpha}{}_{\beta}:={*}(u\wedge\mathcal{D}\Phi^{\alpha}{}_{\beta})$.

Multiplying the equation (3.6) by u from the left externally and using (3.2) one derives the expression for the Lagrange multiplier λ,

$$2nc^{2}\lambda=\varepsilon+p\ . \tag{3.8}$$

As a consequence of the equation (3.2) and the constraints (2.4), (2.5), (2.8), (2.10) one can verify that the Lagrangian density 4-form (3.1) is proportional to the hydrodynamic fluid pressure, $\mathcal{L}_{m}=p\eta$.

4. THE EVOLUTION EQUATIONS OF THE SPIN TENSOR AND THE DILATONIC CHARGE

The variational equation (3.7) represents the evolution equation of the directors. Multiplying the equation (3.7) by $l^{q}_{\beta}\theta^{\alpha}\wedge\ldots$ from the left externally one gets,

$$\frac{1}{2}\dot{S}^{\alpha}{}_{\beta}+\frac{1}{8}\dot{J}\delta^{\alpha}_{\beta}-\chi^{r}S^{\alpha}{}_{r}u_{\beta}-\zeta_{r}S^{r}{}_{\beta}u^{\alpha}=0\ . \tag{4.1}$$

The contraction (4.1) on the indices α and β gives with the help of the Frenkel condition (2.6) the dilatonic charge conservation law,

$$\dot{J}=0\ . \tag{4.2}$$

Contracting (4.1) with u_{α} and then with u^{β} and taking into account (4.2) and the Frenkel condition (2.6), one gets the expressions for the Lagrange multipliers,

$$\zeta_{r}S^{r}{}_{\beta}=-\frac{1}{2c^{2}}\dot{S}^{\gamma}{}_{\beta}u_{\gamma}\ ,\qquad \chi^{r}S^{\alpha}{}_{r}=-\frac{1}{2c^{2}}\dot{S}^{\alpha}{}_{\gamma}u^{\gamma}\ . \tag{4.3}$$

As the consequence of (4.2) and (4.3) the equation (4.1) yields the evolution equation of the spin tensor,

$$\dot{S}^{\alpha}{}_{\beta} + \frac{1}{c^2}\dot{S}^{\alpha}{}_{\gamma}u^{\gamma}u_{\beta} + \frac{1}{c^2}\dot{S}^{\gamma}{}_{\beta}u_{\gamma}u^{\alpha} = 0 \; . \tag{4.4}$$

This equation generalizes the evolution equation of the spin tensor in the Weyssenhoff fluid theory to a Weyl–Cartan space. With the help of the projection tensor (2.7) the equations (4.4) and (4.2) can be represented in the equivalent form,

$$\Pi^{\alpha}_{\sigma}\Pi^{\rho}_{\beta}\dot{J}^{\sigma}{}_{\rho} = 0 \; , \tag{4.5}$$

which is the evolution equation of the total dilaton-spin tensor.

5. THE ENERGY-MOMENTUM TENSOR OF THE PERFECT DILATON-SPIN FLUID

By means of the variational derivatives of the matter Lagrangian density (3.1) one can derive the external matter currents which are the sources of a gravitational field. In case of the perfect dilaton-spin fluid the matter currents are the canonical energy-momentum 3-form Σ_{σ}, the metric stress-energy 4-form $\sigma^{\alpha\beta}$, the dilaton-spin momentum 3-form $\mathcal{J}^{\alpha}{}_{\beta}$.

The variational derivative of the Lagrangian density (3.1) with respect to θ^{σ} yields the canonical energy-momentum 3-form,[6]

$$\Sigma_{\sigma} := \frac{\delta\mathcal{L}_m}{\delta\theta^{\sigma}} = -\varepsilon\eta_{\sigma} + 2\lambda n u_{\sigma}u + 2c^2\lambda n\eta_{\sigma} - n\chi^{r}S^{q}{}_{r}(g_{\sigma\rho}l^{\rho}_{q}u + u_{\sigma}l_{q}) + \frac{1}{2}n\dot{S}^{\rho}{}_{\sigma}\eta_{\rho} \; . \tag{5.1}$$

Using the explicit form of the Lagrange multiplier (3.8), the Frenkel condition (2.6) and the dilatonic charge conservation law (4.2), one gets,

$$\Sigma_{\sigma} = p\eta_{\sigma} + \frac{1}{c^2}(\varepsilon + p)u_{\sigma}u + \frac{1}{2}n\dot{S}^{\rho}{}_{\sigma}\eta_{\rho} - n\chi^{r}S^{\rho}{}_{r}(g_{\sigma\rho}u + l^{q}_{\rho}l_{q}u_{\sigma}) \; . \tag{5.2}$$

On the basis of the evolution equation of the spin tensor (4.4) and with the help of (4.3) the expression (5.2) takes the form,

$$\Sigma_{\sigma} = p\eta_{\sigma} + \frac{1}{c^2}(\varepsilon + p)u_{\sigma}u + \frac{1}{c^2}n g_{\alpha[\sigma}\dot{S}^{\alpha}{}_{\beta]}u^{\beta}u \; . \tag{5.3}$$

After some algebra using the constraints (2.3) and (2.6) one can get that in a Weyl–Cartan space the expression of the canonical energy-momentum 3-form is simplified,

$$\Sigma_\sigma = p\eta_\sigma + \frac{1}{c^2}(\varepsilon + p)u_\sigma u + \frac{1}{c^2}n\dot{S}_{\sigma\rho}u^\rho u \; . \tag{5.4}$$

This expression literally coincides with the expression for the canonical energy-momentum 3-form of the Weyssenhoff perfect spin fluid in a Weyl–Cartan space. It should be mentioned that in case of the dilaton-spin fluid the specific energy density ε in (5.4) contains the energy density of the dilatonic interaction of the fluid.

The metric stress-energy 4-form[6] can be derived in the same way,

$$\begin{aligned} &\sigma^{\alpha\beta} := 2\frac{\delta\mathcal{L}_m}{\delta g_{\alpha\beta}} = T^{\alpha\beta}\eta \; , \\ &T^{\alpha\beta} = pg^{\alpha\beta} + \frac{1}{c^2}(\varepsilon + p)u^\alpha u^\beta + \frac{1}{c^2}n\dot{S}^{(\alpha}{}_\gamma u^{\beta)}u^\gamma \; . \end{aligned} \tag{5.5}$$

The dilaton-spin momentum 3-form can be obtainted in the following way,

$$\mathcal{J}^\alpha{}_\beta := -\frac{\delta\mathcal{L}_m}{\delta\Gamma^\beta{}_\alpha} = \frac{1}{2}n\left(S^\alpha{}_\beta + \frac{1}{4}J\delta^\alpha_\beta\right)u = \mathcal{S}^\alpha{}_\beta + \frac{1}{4}\mathcal{J}\delta^\alpha_\beta \; , \tag{5.6}$$

where the spin momentum 3-form $\mathcal{S}^\alpha{}_\beta = \mathcal{J}^{[\alpha}{}_{\beta]} = (1/2)nS^\alpha{}_\beta u$ and the dilaton current 3-form $\mathcal{J} = \mathcal{J}^\lambda{}_\lambda = (1/2)nJu$ are introduced.

6. CONCLUSIONS

The variational theory of the perfect fluid with intrinsic spin and dilatonic charge has been developed. The essential feature of the constructed variational theory consists in using the Frenkel condition for the spin tensor only but not for the total dilaton-spin tensor. The Lagrangian density of the perfect dilaton-spin fluid has been stated and the equations of motion of the fluid and the evolution equations of the spin tensor and the dilatonic charge have been derived. The expressions of the matter currents of the fluid (the canonical energy-momentum 3-form, the metric stress-energy 4-form and the dilaton-spin momentum 3-form), which are the sources of a gravitational field in the Weyl–Cartan space-time, have been obtained. In the following paper these expressions will be used in deriving the generalized Euler-type hydrodynamic equation of motion of the perfect dilaton-spin fluid and

the equation of motion of a test particle with spin and dilatonic charge in the Weyl–Cartan geometry background.

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